Study materials of Mathematics for class D-III (H), Paper-VI on the topic "Inner Product space", composed by Dr. S. Ahmad, Associate Professor, U.R. College Rosera,

Example 1. Prove that $(u, v) = a_1 + a_2 + b_1 + b_2$ does not define an inner Product in $V_2(R)$.

Soln. We have

(i) $(u, v) = a_1 + a_1 + b_1 + b_2$

$= b_1 + b_2 + a_1 + a_2$

$= (v, u)$

$\therefore (u, v) = (v, u)$ for $V_2(R)$

(ii) $\alpha u + \beta v = \alpha(a_1, a_2) + \beta(b_1, b_2)$

$= (\alpha a_1, \alpha a_2) + (\beta b_1, \beta b_2)$

$= (\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2)$

$\therefore (\alpha u + \beta v, w) = \alpha a_1 + \beta b_1 + \alpha a_2 + \beta b_2 + c_1 + c_2$

But $\alpha(u, w) = \alpha(a_1 + a_2 + c_1 + c_2)$

& $\beta(v, w) = \alpha(b_1 + b_2 + c_1 + c_2)$

clearly $(\alpha u + \beta v, w) \neq \alpha(u, w) + \beta(v, w)$

Hence the given definition of (u, v) does not define an inner product.

Example. Prove that $(u,v) = 2a_1\overline{b_1} + a_1\overline{b_2} + a_2\overline{b_1} + a_2\overline{b_2}$ defines an inner product in $V_2(F)$ and
$(u,v) = 2a_1b_1 + a_1b_2 + a_2b_1 + a_2b_2$
defines an inner product on $V_2(R)$.

Solution. (i) We have

$$\overline{(u,v)} = \overline{2a_1\overline{b_1} + a_1\overline{b_2} + a_2\overline{b_1} + a_2\overline{b_2}}$$

$$= 2\overline{a_1}\,b_1 + \overline{a_1}\,b_2 + \overline{a_2}\,b_1 + \overline{a_2}\,b_2$$

$$= 2b_1\overline{a_1} + b_1\overline{a_2} + b_2\overline{a_1} + b_2\overline{a_2}$$

$$= (v,u)$$

(ii) $\{\alpha u + \beta v\} = \alpha(a_1, a_2) + \beta(b_1, b_2)$

$$= (\alpha a_1, \alpha a_2) + (\beta b_1, \beta b_2)$$

$$= (\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2)$$

$$\therefore (\alpha u + \beta v, w) = 2(\alpha a_1 + \beta b_1)\overline{c_1} + (\alpha a_1 + \beta b_1)\overline{c_2} + (\alpha a_2 + \beta b_2)\overline{c_1} + (\alpha a_2 + \beta b_2)\overline{c_2}$$

by defn.

$$= \alpha(2a_1\overline{c_1} + a_1\overline{c_2} + a_2\overline{c_1} + a_2\overline{c_2}) + \beta(2b_1\overline{c_1} + b_1\overline{c_2} + b_2\overline{c_1} + b_2\overline{c_2})$$

$$= \alpha(u,w) + \beta(v,w).$$

(iii) Again, we have

$$(u,u) = 2a_1\overline{a_1} + a_1\overline{a_2} + a_2\overline{a_1} + a_2\overline{a_2}$$

$$= 2a_1\overline{a_1} + a_2\overline{a_2} + a_1\overline{a_2} + a_2\overline{a_1}$$

$$= 2|a_1|^2 + |a_2|^2 + a_1\overline{a_2} + \overline{a_1\overline{a_2}}$$

$$[\because \overline{a_1\overline{a_2}} = \overline{a_1}\, a_2]$$

$$= 2|a_1|^2 + |a_2|^2 + 2 \text{ real part of } a_1\overline{a_2} \quad \text{--- (1)}$$

Now, let $a_1 = p + iq$ and $a_2 = r + is$ so that $\overline{a_2} = r - is$

$$\therefore a_1\overline{a_2} = (p+iq)(r-is)$$

$$= (pr + qs) + i(qr - ps)$$

$\therefore$ Real part of $a_1\overline{a_2} = pr + qs$

Hence from (1), we have

$$(u,u) = 2(p^2+q^2) + (r^2+s^2) + 2(pr+qs)$$

$$= (p^2+q^2) + (p^2+q^2) + (r^2+s^2) + 2(pr+qs)$$

$$= p^2+q^2 + (p+r)^2 + (q+s)^2 \geq 0$$

Clearly $(u,u) = 0 \Rightarrow p = 0,\ q = 0,\ p+r = 0,\ q+s = 0$

$$\Rightarrow p = 0,\ q = 0,\ r = 0,\ s = 0$$

$$\Rightarrow a_1 = 0,\ a_2 = 0$$

$$\Rightarrow u = (a_1, a_2) = (0,0) = 0$$

Hence the given equation defines an inner product.